

Dynamic Models in Reliability and Survival Analysis

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Abstract

Dynamic models are important and realistic in many settings, notably reliability and survival analysis. Two general classes of dynamic models are described, some probabilistic properties presented, inferential methods indicated, and their applicability discussed.

1 Introduction

Consider a system which is being monitored for the occurrence of events of interest over some observation period. Such a system could be a p -component coherent structure with structure function ϕ in a reliability or engineering setting, where the events of interest are component failures and eventually system failure. Or, the system could be a subject in a biomedical study, and of interest are the successive occurrences of some recurrent event, such as for example hospitalization, onset of depression, etc. Or, the system could be a married couple in a sociological study to examine divorces, and of interest are the recurrences of major disagreements in the marriage.

In developing stochastic models for event occurrences in such systems, dynamic models become highly appropriate and more realistic. In such models the impact of actions or interventions which are undertaken as the monitoring progresses, with such actions possibly dictated by the accrued history of the system, can be incorporated in the model. Dynamic models could also incorporate the possible effects of increasing number of event occurrences, and take into account the impact of possibly time-varying covariate processes. For example, when dealing with coherent structures in reliability and engineering settings, component failures may have the effect of increasing the load on the remaining functioning components arising from the change in the effective structure function, hence the failure intensities of components are best specified through dynamic models. In biomedical settings and reliability settings, when an event occurs, certain interventions are performed, and such actions have the effect of changing the intensities of the next event occurrence.

Dynamic models are best specified through failure intensities, and so are often stated in terms of hazard rate functions. In contrast to the usual static modelling approach, which when specifying a failure model at some time s , asks the following question at the time origin: “What is the probability that an event will occur in the infinitesimal time interval $[s, s + ds)$?” in dynamic modelling, the underlying modelling approach is conditional in that one instead asks the question: “Given the history of the system just before time s , what is the conditional probability that an event will occur in the infinitesimal time interval $[s, s + ds)$?” Because the modelling approach is conditional, then it is able to seamlessly incorporate in the model the impact of performed actions and interventions, as well as the situational and environmental changes, that dynamically occur during the monitoring of the system.

Many works in the reliability and engineering settings dealing with dynamic models have dealt with the modelling aspect and the determination of the stochastic and probabilistic properties of such models. There has been a dearth of work dealing with statistical inference issues for such dynamic models. In the survival analysis setting where dynamic models are typically associated with biomedical studies and clinical trials, there has been work dealing with inference issues for such models, though for general dynamic models, inference procedures are still incomplete. In this talk, aside from discussing dynamic models and

their properties, focus will also be on methods for making inference about the model parameters, with the inference to be based on data arising from the monitoring of a sample of study systems or units.

We outline the major portions of this abstract. In Section 2 we present the mathematical setting which will facilitate the formal description of dynamic models, and describe three specific dynamic models. In Section 3 we provide some properties of the dynamic models described in Section 2, and in Section 4 we indicate inference issues regarding dynamic model parameters. The issue of the practical applicability of the models will be mentioned in Section 5, and finally in Section 6 we indicate open research problems for these dynamic models.

2 Dynamic Models

To formally describe the dynamic models of interest, given a system or unit under study, let $\mathcal{T} = [0, \tau]$ be the monitoring period, with τ possibly random. Define the processes $\{(N^\dagger(s), Y^\dagger(s)) : s \in \mathcal{T}\}$ according to $N^\dagger(s)$ = number of events that occurred in $[0, s]$, and $Y^\dagger(s) = I\{\text{the system is under observation at } s\}$, where $I\{A\}$ is the indicator function of event A . We also define on an appropriate probability space (Ω, \mathcal{F}, P) a filtration $\{\mathcal{F}_s : 0 \leq s \leq \tau\}$, such that \mathcal{F}_s represents the σ -field containing all information about the system that have accrued over the time period $[0, s]$. In particular, N^\dagger and Y^\dagger are adapted to this filtration. A dynamic model is specified by providing the intensity process of N^\dagger defined for every $s \in \mathcal{T}$ via

$$\alpha(s) = \lim_{h \downarrow 0} \frac{1}{h} P \{N^\dagger((s+h)-) - N^\dagger(s-) \geq 1 | \mathcal{F}_{s-}\}. \quad (1)$$

We describe two general classes of dynamic models. The first one, which is relevant for coherent systems in reliability and engineering, was introduced in Hollander and Peña (1995); while the second one, which perhaps is more relevant in biomedical and public health settings, was introduced in Peña and Hollander (2004).

Consider a coherent system with p components and structure function ϕ . Let $\mathcal{Z}_p = \{1, 2, \dots, p\}$, and denote by \mathcal{P} the power set of \mathcal{Z}_p . Let $\mathcal{K}_\phi \subseteq \mathcal{P}$ be the collection of minimal cut sets of ϕ . A set $J \in \mathcal{P}$ is defined to be ϕ -absorbing if there exists a $K \in \mathcal{K}_\phi$ with $K \subseteq J$. Let \mathcal{Q}_ϕ be the collection of ϕ -absorbing sets of ϕ , and by $\mathcal{Q}_\phi^0 = \mathcal{P} \setminus \mathcal{Q}_\phi$ the collection of ϕ -non-absorbing sets of ϕ . We now describe the first dynamic model. Let $\lambda_0(\cdot)$ be a hazard rate function, and for each $J \in \mathcal{Q}_\phi^0$, let $\{\alpha_i[J], i \in I^c\}$ be a set of non-negative real numbers. For each $s \geq 0$, denote by $F(s)$ the set of component indices which are non-functioning at time $s-$. With $Y^\dagger(s) = I\{\tau \geq s\}$, the intensity process of the dynamic model is specified according to

$$\alpha(s) = Y^\dagger(s) \left[\sum_{J \in \mathcal{Q}_\phi^0} I\{F(s) = J\} \sum_{j \in J^c} \alpha_j[J] \right] \lambda_0(s). \quad (2)$$

This model is a special case of the general model introduced in Hollander and Peña (1995). If the system is a p -component parallel system so $\mathcal{Q}_\phi = \mathcal{Z}_p$ and for any state vector $(y_1, y_2, \dots, y_p) \in \{0, 1\}^p$, $\phi(y_1, y_2, \dots, y_p) = \bigvee_{j=1}^p y_j$, and if $\alpha_j[J] = \gamma_{|J|}$ where $|J|$ is the cardinality of set J and $\{\gamma_j \equiv \gamma[j] : j = 0, 1, \dots, p-1\}$ are non-negative reals with $\gamma_0 \equiv 1$, then (2) simplifies to

$$\alpha(s) = Y^\dagger(s) [p - N^\dagger(s-)] \gamma[N^\dagger(s-)] \lambda_0(s), \quad (3)$$

which is the equal load-sharing model for a parallel system considered in Kvam and Peña (2004).

Next we describe a general dynamic model for recurrent events which takes into account the impact of performed interventions after each event occurrence, the effects of accumulating event occurrences and relevant covariate processes, and the effect of an unobserved latent variable, called a frailty, which induces association among the inter-event times. This model was proposed in Peña and Hollander (2004), and further studied in Peña, Slate and Gonzalez (2003). To specify the intensity process for this model, we suppose that a vector of predictable covariate processes $\{\mathbf{X}(s) : s \geq 0\}$ is observed, and we also require an observable and

predictable effective age process $\{\mathcal{E}(s) : s \geq 0\}$, which is possibly specified dynamically. Conditional on the frailty variable Z , which is assumed to follow a distribution $H(\cdot|\xi)$, the intensity process is given by

$$\alpha(s|Z) = Z Y^\dagger(s) \lambda_0[\mathcal{E}(s)] \rho[N^\dagger(s-); \alpha] \psi[\beta' \mathbf{X}(s)], \quad (4)$$

where $\lambda_0(\cdot)$ is a hazard rate function, $\rho(\cdot; \alpha)$ is a non-negative function with $\rho(0; \alpha) = 1$, and $\psi(\cdot)$ is a non-negative link function. In this model, the effective age process encodes the impact of performed interventions after each event occurrence. If minimal repair or intervention is performed after each event occurrence, this effective age process takes the form $\mathcal{E}(s) = s$, whereas if perfect repair or intervention is performed after each event occurrence, then this is the backward recurrence time given by $\mathcal{E}(s) = s - S_{N^\dagger(s-)}$, where $0 \equiv S_0 < S_1 < S_2 < \dots$ are the successive calendar times of event occurrences. Many other forms of $\mathcal{E}(\cdot)$ are possible such as that induced by the minimal repair model of Brown and Proschan (1983) and Block, Borges and Savits (1985). The effect of accumulating event occurrences is contained in the $\rho(\cdot; \alpha)$ function, and a simple form for this is $\rho(k; \alpha) = \alpha^k$; whereas the covariate effect is encoded in the link function $\psi(\cdot)$, which is usually taken to be $\psi(v) = \exp(v)$. The frailty distribution $H(\cdot; \xi)$ could have many forms, but in many cases it is assumed to be a gamma distribution with mean 1 and variance $1/\xi$. The baseline hazard rate function $\lambda_0(\cdot)$ could either be parametrically specified, or could be nonparametrically specified. The latter may be more appropriate in biomedical and public health applications, whereas the former is more appropriate in reliability and engineering applications. As discussed in Peña and Hollander (2004) and in Peña, Slate and Gonzalez (2003), the class of models specified by (4) includes as special cases many existing models currently in use in reliability and survival analysis.

3 Some Properties

Probabilistic properties of dynamically-specified models are certainly harder to obtain due to the changing intensities as time evolves. Nevertheless, in some special cases, concrete results are possible. Just to provide a flavor for such results, consider a p -component parallel system with intensity process specified in (2). We present the distribution of the time to the k th event as obtained in Hollander and Peña (1995). To state this result, we need to introduce notation. For a collection $\mathbf{a} = \{a_i : i \in \mathcal{C}\}$ of distinct real numbers, define

$$a_\bullet = \sum_{i \in \mathcal{C}} a_i \quad \text{and} \quad \rho_i(a_j; \mathcal{C}) = \prod_{j \in \mathcal{C}; j \neq i} \frac{a_j}{a_j - a_i}, \forall i \in \mathcal{C}.$$

In the notation $\rho_i(a_j; \mathcal{C})$, \mathcal{C} denotes the set of possible values of the index j . We also utilize the notation $\mathcal{C}_k = \{0, 1, \dots, k\}$ for $k = 0, 1, 2, \dots$. Following earlier notation, let S_k be the time of the k th event for this system, which corresponds to the k th component failure. The following result is a re-statement of Theorem 5.1 in Hollander and Peña (1995); in there, more general distributional results from which the result below was derived, as well as specific results for series-parallel systems, were also presented.

Theorem 1 *For a p -component parallel system following a dynamic model with intensity process in (2), if the collection $\{\alpha_\bullet[J] \equiv \sum_{j \in J^c} \alpha_j[J] : J \subset \mathcal{Z}_p\}$ satisfies the condition that $|J| = k \Rightarrow \alpha_\bullet[J] = \alpha_k$, ($k = 0, 1, \dots, p$), with $\alpha_k \neq \alpha_l$ whenever $k \neq l$, then for $k = 1, 2, \dots, p$ and with $\Lambda_0(s) = \int_0^s \lambda_0(t) dt$,*

$$P\{S_k > s\} = \sum_{i=0}^{k-1} \rho_i(\alpha_j; \mathcal{C}_{k-1}) \exp\{-\alpha_i \Lambda_0(s)\}.$$

4 Inference Methods

Suppose that a sample of n systems or units governed by the dynamic models described above are monitored, with the i th system followed over $[0, \tau_i]$. Then it becomes of interest to make inference about the unknown model parameters. Having estimates of these model parameters will enable us to perform predictions as well as aid in making practical decisions such as, for example, performing preventive maintenance or doing

some interventions. Inference methods for these dynamic models become more elaborate and complicated, especially if the baseline hazard rate function $\lambda_0(\cdot)$ is nonparametrically specified. The major tools that enable us to do inference is the construction of the likelihood function via Jacod's (1975) formula (see also Andersen, Borgan, Gill and Keiding (1993)); the fact that with respect to calendar time there is a martingale structure to $\{M^\dagger(s) = N^\dagger(s) - \int_0^s \alpha(v)dv : s \geq 0\}$; and, to deal with the computational complexity, through the use of the EM algorithm of Dempster, Laird and Rubin (1977).

In Kvam and Peña (2004), the estimation of the load-share parameters $\{\gamma_j : j = 0, 1, \dots, p-1\}$, the baseline hazard $\Lambda_0 = \int \lambda_0$, and the associated baseline survivor function $S_0 = \prod [1 - d\Lambda_0]$ were developed for the model in (3). Asymptotic properties of the estimators were also obtained. For the general recurrent event model specified in (4), estimation procedures for the model parameters, which are ξ in the frailty distribution, α in the $\rho(k; \alpha)$ component, β in the link function, and for Λ_0 and S_0 , were developed in Peña, Slate and Gonzalez (2003). Properties of the estimators were ascertained through computer simulation studies. In the talk, some of these inference methods developed in these papers, as well as in other papers, will be discussed.

5 Applicability of Models

The applicability of these dynamic models is still in its infancy. Nevertheless, by virtue of the fact that they are more appropriate models of real situations, their practical promise is quite appealing. In the talk, certain applications of these dynamic models to real data sets will be illustrated. Through these illustrations the importance, as well as existing limitations, of these dynamic models will be pinpointed.

6 Some Open Problems

Because these dynamic models are still new and currently the subject of active research, many open research problems abound. Some of these open research problems will be described and posed in the talk.

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